Ma2a Practical – Recitation 6

November 8, 2024

Exercise 1. (Chapter 3.3, Exercise 34: Euler Equations.) An equation of the form

$$t^2 \frac{d^2y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = 0, \quad t > 0,$$

where α and β are real constants, is called an Euler equation.

1. Let $x = \ln t$ and calculate dy/dt and d^2y/dt^2 in terms of dy/dx and d^2y/dx^2.

2. Use the results of the previous question to transform the equation into:

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0.$$

   If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of the new equation, then $y_1(\ln t)$ and $y_2(\ln t)$ form a fundamental set of solutions of the original equation.

Exercise 2. (Chapter 7.1, Exercise 7) Systems of first order equations can sometimes be transformed into a single equation of higher order. Consider the system:

$$x_1' = -2x_1 + x_2, \quad x_2' = x_1 - 2x_2.$$

1. Solve the first equation for x_2 and substitute into the second equation, thereby obtaining a second order equation for x_1. Solve this equation for x_1 and then determine x_2 also.

2. Find the solution of the given system that also satisfies the initial conditions $x_1(0) = 0$, $x_2(0) = 3$.
   We denote by $\gamma\colon t \mapsto (x_1(t), x_2(t))$ the associated parametric curve in the x_1x_2-plane.

3. Study the existence of an asymptotic to γ as $t \to +\infty$, and of points with horizontal or vertical tangents. Use these observations to plot the solution curve in the x_1x_2-plane.

Exercise 3. Solve the (Clairaut) equation:

$$y(x) = xy' + (y')^2. \tag{1}$$

Exercise 4. Find a curve that passes through the point $(1, 1)$ and perpendicular to all curves of the family:

$$\Gamma_C : x^4 + y^4 = C \quad (C \in \mathbb{R}). \tag{2}$$

Find all smooth curves that intersect any curve of the family $(\Gamma_C)_{C\in\mathbb{R}}$ at the right angle.

Solution 1

1. Considering the variable change $x = \ln t$, we have $t = e^x$. At the beginning, we view $y = y(t)$ as a function of t. Right now, we consider $y = y(e^x)$ as a function of x. Then the derivative $\frac{dy}{dx} = y'(e^x)e^x = y'(t)t = t\frac{dy}{dt}$. Besides, the second order derivative $\frac{d^2y}{dx^2} = \frac{dy'(e^x)e^x}{dx} = y''(e^x)e^{2x} + y'(e^x)e^x = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$. Thus, in terms of dy/dx and d^2y/dx^2, we have $dy/dt = t^{-1}dy/dx = e^{-x}dy/dx$ and $d^2y/dt^2 = t^{-2}(d^2y/dx^2 - dy/dx) = e^{-2x}(d^2y/dx^2 - dy/dx)$

2. With the result in the previous equation, we know that $\frac{d^2y}{dx^2} - \frac{dy}{dx} = t^2\frac{d^2y}{dt^2}$ and $\frac{dy}{dx} = t\frac{dy}{dt}$. Thus, under the change of variable, the equation could be rewritten as
$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0.$$
This is a homogeneous second order linear differential equation. We know that if r_1 and r_2 are two roots of the characteristic polynomial $r^2 + (\alpha - 1)r + \beta = 0$. Then the general solution is
$$y(x) = \begin{cases} Ae^{r_1x} + Be^{r_2x}, & r_1 \neq r_2 \\ Ae^{r_1x} + Bxe^{r_1x}, & r_1 = r_2 \end{cases}$$
Rechange the variable $x = \ln t$, we have the general solution of $y(t)$ is given by
$$y(t) = \begin{cases} At^{r_1} + Bt^{r_2}, & r_1 \neq r_2 \\ At^{r_1} + Bt^{r_1}\ln t, & r_1 = r_2 \end{cases}$$
Thus, if $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of the new equation, then $y_1(\ln t)$ and $y_2(\ln t)$ form a fundamental set of solutions of the original equation.

Solution 2:

1. From the first equation,
$$x_2 = x_1' + 2x_1 \qquad (\star)$$
and differentiating this, we get
$$x_2' = x_1'' + 2x_1'.$$
Now substitute these expressions for x_2, x_2' into the second equation:
$$x_1'' + 2x_1' = x_1 - 2(x_1' + 2x_1) = -2x_1' - 3x_1.$$
After moving all the terms to one side, this is
$$x_1'' + 4x_1' + 3x_1 = 0.$$

This is a second-order homogeneous linear ODE with constant coefficients. Its characteristic polynomial is $r^2+4r+3=(r+1)(r+3)=0$ with roots $r_1=-1$, $r_2=-3$, so the general solution is

$$\boxed{x_1(t)=c_1e^{-t}+c_2e^{-3t},}$$

hence

$$x_1'(t)=-c_te^{-t}-3c_2e^{-3t}.$$

Putting this back into $(\star)$, we get

$$x_2(t)=-c_te^{-t}-3c_2e^{-3t}+2(c_1e^{-t}+c_2e^{-3t})=\boxed{c_1e^{-t}-c_2e^{-3t}}$$

2. We have the system of equations:

$$\begin{cases} x_1(0)=c_1+c_2=0 \\ x_2(0)=c_1-c_2=3 \end{cases}$$

which has the unique solution $c_1=3/2$, $c_2=-3/2$. Now if we write $(x_1(t),x_2(t))$ as a vector,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}=\begin{pmatrix} \frac{3}{2}e^{-t}-\frac{3}{2}e^{-3t} \\ \frac{3}{2}e^{-t}+\frac{3}{2}e^{-3t} \end{pmatrix}=\frac{3}{2}e^{-2t}\begin{pmatrix} e^{t}-e^{-t} \\ e^{t}+e^{-t} \end{pmatrix}=3e^{-2t}\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}$$

3. As $t\to\infty$,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}=\frac{3}{2}e^{-2t}\begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}$$

This is the asymptotic behavior. The derivatives are

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}=\frac{3}{2}e^{-2t}\begin{pmatrix} -e^{-t}+3e^{-3t} \\ -e^{-t}-3e^{-3t} \end{pmatrix}$$

A horizontal tangent line is characterized by $x_2'(t)=0$. This equation is equivalent to $e^{2t}=-3$, so it does not have solutions: there are no such tangents.

A vertical tangent line is characterized by $x_1'(t)=0$, which is equivalent to the equation $e^{2t}=3$, and has the unique solution $t=\ln\frac{3}{2}$.

The solution curve and its asymptote are plotted on Figure 1.

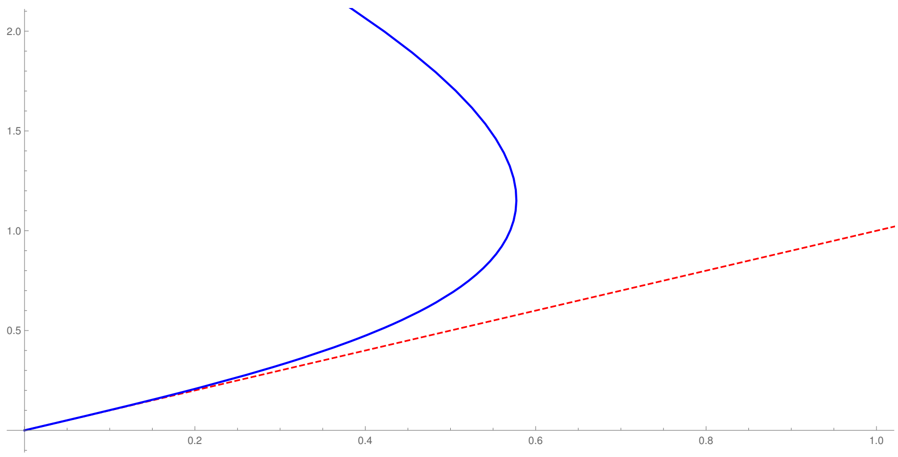

Figure 1: Plot of the parametric curve γ in Problem 2 (blue), and asymptote as $t \to +\infty$ (red dashed).

Solution 3:

We consider the general Clairaut's equation which is:

$$y = xy' + f(y')$$

In our case, $f(y') = (y')^2$

To solve Clairaut's equation, one differentiates with respect to x, yielding

$$y' = y' + xy'' + f'(y')\, y''$$

so

$$\left[x + f'(y')\right] y'' = 0$$

Hence, either

$$y'' = 0$$

or

$$x + f'(y') = x + 2y' = 0$$

In the former case, $C = y'$ for some constant C. Substituting this into Clairaut's equation, one obtains the family of straight line functions given by

$$y(x) = Cx + f(C) = Cx + C^2$$

the so-called general solution of Clairaut's equation.

The latter case, $y = -\frac{x^2}{4} + C$, substituting back into the Clairaut's equation, we see $C = 0$

$$y = -\frac{x^2}{4}$$

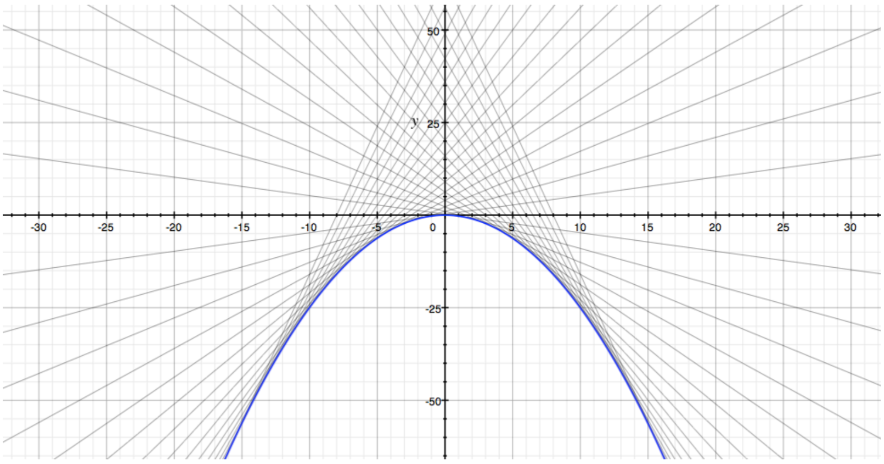

Figure 2: Solution curves for the Clairaut equation. The general solutions are the gray lines, the singular solution is the blue curve.

defines only one solution $y(x)$, the so-called singular solution, whose graph is the envelope of the graphs of the general solutions (see Figure 2).

Solution 4

Fix one $C \in \mathbb{R}$, the tangent direction at (x, y) is given by $(1, dy/dx)$. To find it, we take derivatives of $x^4 + y^4 = C$ with respect to x

$$y' = -\frac{x^3}{y^3}.$$

So for the curve perpendicular to it, the tangent direction needs to be $(1, \frac{y^3}{x^3})$. We at a point (x, y) it satisfies the equation

$$\frac{dy}{dx} = \frac{y^3}{x^3}.$$

Integrate to get, $y^{-2} = x^{-2} + C$ and it passes through the point $(1, 1)$ implies $C = 0$ and $y = x$ or $y = -x$.

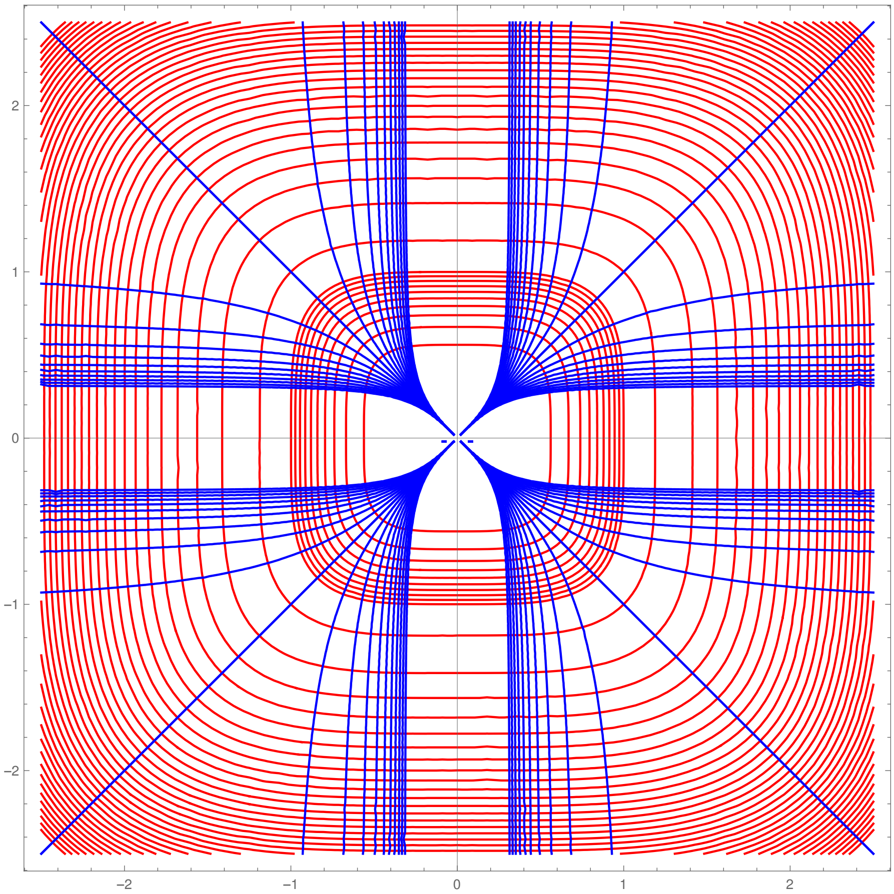

Figure 3: The family of curves $\{\Gamma_C\}_{C\in\mathbb{R}}$ (in red) and the orthogonal family of curves (in blue).